Lesson 2. Introduction to Optimization Modeling

1 The Farmer Jones problem

Farmer Jones decides to supplement her income by baking and selling two types of cakes, chocolate and vanilla. Each chocolate cake sold gives a profit of $3, and the profit on each vanilla cake sold is $4. Each chocolate cake uses 4 eggs and 4 pounds of flour, while each vanilla cake uses 2 eggs and 6 pounds of flour. If Farmer Jones has only 32 eggs and 48 pounds of flour available, how many of each type of cake should Farmer Jones bake in order to maximize her profit? (For now, assume all cakes baked are sold, and fractional cakes are OK.)

- Let C be a variable that represents the number of chocolate cakes that Farmer Jones bakes

- Similarly, let V be a variable that represents the number of vanilla cakes that Farmer Jones bakes

- What are some allowable values for C and V? What is the corresponding profit?

	Choc.	Vanilla	Avail
Eggs	4	2	32
Flour	4	6	48
Profit	3	4	

C	V	profit
0	0	0
0	8	32
6	4	34
3	6	33
8	0	24

- Can we generalize this? In particular: *feasible solutions* | *values*

    ○ Can we describe the profit in terms of C and V?

    ○ Can we describe the set of all possible values for C and V?

- In other words, can we describe Farmer Jones's problem as an **optimization model**?

2   Formulating an optimization model

- An **optimization model** or **mathematical program** consists of:
  1. **Constants**: numbers that are fixed
     - **Parameters**: constants that are represented by symbols
  2. **Decision variables**: variables that represent decisions to be made
  3. **Objective function**: function of the decision variables to be maximized or minimized
  4. **Constraints**
     a. **Variable bounds**: specify the values for which the decision variables have meaning
     b. **General constraints**: specify all other restrictions, requirements, and interactions that could limit the values of the decision variables

- We end up with something that looks like this:

$$
\begin{aligned}
&\text{(constant/parameter definitions)}\\
&\text{(decision variable definitions)}\\
\text{maximize/minimize} \quad &\text{(objective function)}\\
\text{subject to} \quad &\text{(general constraints)}\\
&\text{(variable bounds)}
\end{aligned}
$$

- Let's write an optimization model for Farmer Jones's model

- What are the constants and parameters?

profit per cake type, # eggs + flour required for each cake type,
eggs + flour available

- Decision variables:

C = # chocolate cakes to bake
V = # vanilla cakes to bake.

- Objective function:

maximize $3C + 4V$

- Constraints:

$$
\begin{aligned}
4C + 2V &\le 32 \qquad \text{(eggs)}\\
4C + 6V &\le 48 \qquad \text{(flour)}\\
C &\ge 0 \qquad\\
V &\ge 0 \qquad \text{(nonnegativity)}
\end{aligned}
$$

- Note that the constraints must

  - permit the decision variables to take on all allowable values, e.g. the ones we found above
  - prevent the decision variables from taking on all unallowable values, e.g. $C = 1000$, $V = 1000$

3 Solutions and values of optimization models

- A **feasible solution** to an optimization model is a choice of values for the decision variables that satisfies all constraints

- The **feasible region** of an optimization model is the collection of all feasible solutions to the model

- The **value** of a feasible solution is its objective function value

- An **optimal solution** to an optimization model is a feasible solution whose value is as good as the value of all other feasible solutions

- The **optimal value** of an optimization model is the value of an optimal solution

Example 1. Using trial-and-error, try to find an optimal solution to the Farmer Jones optimization model. In other words, find a feasible solution with the highest value.

> optimal solution: $C = 6$, $V = 4$
> optimal value: 34

4 Classification of optimization models

- Based on characteristics of

  - decision variables
  - constraints
  - objective function

- Decision variables can be continuous or integral

  - **Continuous**: can take on any value in a specified interval, e.g. $[0, +\infty)$
  - **Integral** (also **discrete**): restricted to a specified interval of integers, e.g. $\{0, 1\}$

- Functions can be linear or nonlinear

  - A function $f(x_1, \ldots, x_n)$ is **linear** if it is a constant-weighted sum of $x_1, \ldots, x_n$; i.e.

$$f(x_1, \ldots, x_n) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$$

  where $c_1, \ldots, c_n$ are constants
  - Otherwise, a function is **nonlinear**

- Are these functions linear or nonlinear?

  1   **2**

  - $f(x_1, x_2, x_3) = 9x_1 - 17x_3$

linear

  - $f(x_1, x_2, x_3) = \dfrac{5}{x_1} + 3x_2 - 6x_3$

nonlinear

  - $f(x_1, x_2, x_3) = \dfrac{x_1 - x_2}{x_2 + x_3}$

nonlinear

  - $f(x_1, x_2, x_3) = x_1 x_2 + 3x_3$

nonlinear

- Constraints can be linear or nonlinear

  - A constraint can be written in the form

$$g(x_1, \ldots, x_n) \left\{ \begin{array}{c} \leq \\ = \\ \geq \end{array} \right\} b \qquad (*)$$

    where $g(x_1, \ldots, x_n)$ is a function of decision variables $x_1, \ldots, x_n$ and b is a specified constant

  - Constraint $(*)$ is **linear** if $g(x_1, \ldots, x_n)$ is linear and **nonlinear** otherwise

- Strict inequalities ($<$ or $>$) are <u>not</u> allowed in the optimization models we study

- An optimization model is a **linear program (LP)** if

  - the decision variables are <u>continuous</u>
  - the objective function is <u>linear</u>, and
  - the constraints are <u>linear</u>

- Are these optimization models linear programs?

  - max $\quad 3z_1 + 14z_2 + 7z_3$
    s.t. $\quad 10z_1 + 5z_2 \leq 25 - 18z_3$
    $\qquad z_1 \geq 0, z_2 \geq 0, z_3 \geq 0$

Yes

  - min $\quad 3w_1 + 14w_2 - w_3$
    s.t. $\quad 3w_1 + w_2 \leq 1$
    $\qquad w_1 w_2 w_3 = 1$
    $\qquad w_1 + 2w_2 + w_3 = 10$
    $\qquad w_1 \geq 0, w_3 \geq 0$
    $\qquad w_1$ integer

No

  - Farmer Jones's model

Yes

- There are other types of optimization models: e.g. nonlinear programs, integer programs

- <u>This semester, we will focus on linear programs</u>